

# Finding formulas using multipliers with inverse square potential on $\mathbb{R}^+$

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**Abstract** In this work we give explicit formulas for the Schwartz kernels of the weighted heat, the weighted wave, the weighted resolvent and the weighted generalized resolvent kernels with inverse square potential on  $\mathbb{R}_+^*$ . By using the integral transforms connecting these kernels we obtain old and new formulas involving Bessel and hypergeometric functions.

**Key words** : Weighted heat kernel, Weighted wave and Weighted resolvent kernels, Two variables hypergeometric functions.

## 1-Introduction

The inverse square potential is an interesting potential which arises in several contexts, one of them being the Schrödinger equation in non relativistic quantum mechanics. For example the Hamiltonian for a spin zero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential [4]. The first aim of this paper is to give explicit formulas for the Schwartz kernels of the following multipliers of the Schrödinger operator with inverse square potential called here respectively the weighted heat, weighted wave, weighted resolvent and weighted generalised resolvent operators with inverse square potential on  $\mathbb{R}_+$ .

$$H_\nu^p(t) := e^{tL_\nu}(\sqrt{-L_\nu})^p \quad t > 0 \quad (1.1)$$

$$W_\nu^p(t) := \frac{\sin t\sqrt{-L_\nu}}{\sqrt{-L_\nu}}(\sqrt{-L_\nu})^p \quad t > 0 \quad (1.2)$$

$$R_\nu^p(\lambda) := (L_\nu + \lambda^2)^{-1}(\sqrt{-L_\nu})^p \quad \text{Im}\lambda^2 > 0 \quad (1.3)$$

$$R_\nu^{\mu,p}(\lambda) := (L_\nu + \lambda^2)^{-1-\mu}(\sqrt{-L_\nu})^p \quad \text{Im}\lambda^2 > 0 \quad (1.4)$$

where

$$L_\nu = \frac{\partial^2}{\partial x^2} + \frac{1/4 - \nu^2}{x^2}, \quad \nu \in \mathbb{R} \quad (1.5)$$

is the Schrödinger operator with inverse square potential on  $\mathbb{R}^+$ .

Note that we can define  $\phi(\sqrt{-L_\nu})$  for  $\phi$  a well behaved Borel function by using the Hankel transform (see Proposition 1.1).

Each of these families of multipliers are related to the other by means of various integral transforms. For example using the Laplace transform, one can express all of the above multipliers in terms of the weighted heat operator. Using also Fourier transform one can write general multipliers as integral combinations of the weighted Schrödinger or the weighted wave kernels.

The second aim of this paper is by using the integral transforms connecting these Schwartz kernels we obtain old and new formulas relating Bessel and hypergeometric functions.

First of all we recall the following formulas for the classical heat ([2], p. 68), wave ([16], p.132 – 133) and the resolvent ([11]) kernels with inverse square potential

$$H_\nu^0(t, x, x') = \frac{(xx')^{1/2}}{2t} e^{-\frac{(x^2+x'^2)}{4t}} I_\nu\left(\frac{xx'}{2t}\right) \quad (1.6)$$

$$W_\nu^0(t, x, x') = \begin{cases} 0 & x' > t + x \\ \frac{1}{2} P_{\nu-1/2}\left(\frac{x^2+x'^2-t^2}{2xx'}\right) & t - x < x' < t + x \\ \frac{\cos \pi \nu}{\pi} Q_{\nu-1/2}\left(\frac{t^2-x^2-x'^2}{2xx'}\right) & x' < t - x \end{cases} \quad (1.7)$$

$$R_\nu^0(\lambda, x, x') = \frac{i\pi}{2} \sqrt{xx'} \begin{cases} J_\nu(\lambda x) H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x') H_\nu^{(1)}(\lambda x) & x > x' \end{cases} \quad (1.8)$$

where  $I_\nu$ ,  $J_\nu$ ,  $H_\nu^{(1)}$  are respectively the first kind modified Bessel, the first kind Bessel, the first kind Hankel (see [5], [12]) and the associated Legendre functions of the first and second kind  $P_\nu^\mu$  and  $Q_\nu^\mu$  are given in terms of the Gauss hypergeometric function as ([5], 122):

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu; \frac{1-z}{2}\right) \quad (1.9)$$

$$Q_\nu^\mu(z) = \frac{\sqrt{\pi} e^{i\mu\pi} \Gamma(\mu+\nu+1/2)}{2^{-\nu-1} \Gamma(\mu+\nu+1)} z^{-\mu-\nu-1} (z^2-1)^{\mu/2} F\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1, \nu+3/2, \frac{1}{z^2}\right) \quad (1.10)$$

The Gauss hypergeometric function is defined by ([12], p.238)

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1, \quad (1.11)$$

where as usual  $(a)_n$  is the Pochhammer symbol defined by  $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$  and  $\Gamma$  is the classical Euler function.

The end of this section is devoted to the preliminaries on the Hankel transform on  $\mathbb{R}^+$  and formulas evaluating the Hankel transform of some functions.

For  $\nu > -1$  the Hankel transform of order  $\nu$  for a function  $f \in C_0^\infty(\mathbb{R}^+)$  is defined by the integral

$$(H_\nu f)(\omega) = \int_0^\infty (x\omega)^{1/2} J_\nu(x\omega) f(x) dx \quad (1.12)$$

where  $J_\nu$  is the first order Bessel function of order  $\nu$ .

Recall the following results on the Hankel transform:

**Proposition 1.1** ([15]) For  $\nu > -1$ , we have

- i)  $H_\nu^2 = 1$
- ii)  $H_\nu$  is self adjoint
- iii)  $H_\nu$  is an  $L^2$  isometry
- iv)  $H_\nu L_\nu = -\omega^2 H_\nu$

For more informations on the Hankel transform the reader can consults the nice book by Davies [3].

**Proposition 1.2** For  $\nu > -1$ , the Schwartz kernel of the operator  $\phi(\sqrt{-L_\nu})$  is given at last formally by

$$K_\nu(\phi, x, x') = (xx')^{1/2} \int_0^\infty J_\nu(\omega x) J_\nu(\omega x') \phi(\omega) \omega d\omega \quad (1.13)$$

The proof of this proposition uses essentially the proposition 1.1 and in consequence is left to the reader.

We recall also the following formulas evaluating some Hankel transforms ([6]p.187 and [10]p.685)

$$\int_0^\infty e^{-pt} f(t) dt = \frac{\Gamma(\nu + M)}{\Gamma(2\mu_1 + 1) \Gamma(2\mu_2 + 1)} p^{-\nu-M} a_1^{\mu_1} a_2^{\mu_2} \Psi_2(\nu + M, 2\mu_1 + 1, 2\mu_2 + 1, a_1/p, a_2/p) \quad (1.14)$$

$f(t) = t^{\nu-1} J_{2\mu_1}(2(a_1 t)^{1/2}) J_{2\mu_2}(2(a_2 t)^{1/2})$ ,  $M = \mu_1 + \mu_2$ ,  $\text{Re}(\nu + M) > 0$ , and  $\Psi$  is the two variables confluent hypergéométric function ([5], p.225):

$$\Psi_2(a; c, c', x, y) = \sum_{n, m \geq 0} \frac{(a)_{n+m}}{(c)_m (c')_n m! n!} x^m y^n \quad (1.15)$$

$$\int_0^\infty x^{\rho-1} J_\lambda(ax) J_\mu(bx) J_\eta(cx) dx = \frac{2^{\rho-1} \Gamma(\beta) a^\lambda b^\mu}{\Gamma(\lambda+1) \Gamma(\mu+1) \Gamma(1-\alpha) c^{\lambda+\mu+\rho}} F_4(\alpha, \beta; \lambda+1; \mu+1, a^2/c^2, b^2/c^2) \quad (1.16)$$

where  $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)$ ;  $\gamma, \gamma' \neq 0, -1, -2, \dots$ ,  $\alpha = (\lambda + \mu + \rho - \eta)/2$ ,  $\beta = (\lambda + \mu + \rho + \eta)/2$ ,  $\text{Re} \beta > 0$ ,  $\text{Re} \rho < 5/2$ ,  $a > 0, b > 0, c > 0$ ,  $c > a + b$ .

$F_4$  is the two variables Appell hypergeometric function ([5], p.224) given for  $|\sqrt{x}| + |\sqrt{y}| < 1$  by:

$$F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n \quad (1.17)$$

Finally, recall the following asymptotic formulas for the Bessel functions ([12], p.134)

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(1+\nu)}; x \rightarrow 0; J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - (\nu\pi/2) - (\pi/4)); x \rightarrow \infty \quad (1.18)$$

We end this section by mentioning the organization of this paper. An explicit expression of the Schwartz integral kernels of the weighted heat evolution operator  $e^{tL_\nu}(\sqrt{-L_\nu})^p$  and the weighted Schrödinger evolution operator  $e^{itL_\nu}(\sqrt{-L_\nu})^p$  will be given in the next section 2. In section 3 we will obtain a closed form of the Schwartz integral kernel  $W_\nu^p(t, x, x')$  of the weighted wave evolution operator for  $x + x' < t$  and  $\text{Re} p < \frac{3-n}{2} \frac{\sin t \sqrt{-L_\nu}}{\sqrt{-L_\nu}} (\sqrt{-L_\nu})^p$ . Sections 4 and 5 are devoted to the Schwartz integral kernels of the the weighted and weighted generalized resolvents operator with inverse square potential on  $\mathbb{R}^+$  Othe applications of our results are given in section 6

## 2- Weighted Heat evolution operator with inverse square potential on $\mathbb{R}^+$

In this section explicit formulas for the Schwartz integral kernels of the weighted heat and Schrödinger evolution operators  $e^{tL_\nu}(\sqrt{-L_\nu})^p$  and  $e^{itL_\nu}(\sqrt{-L_\nu})^p$  are given in explicit forms.

**Theorem 2.1** The Schwartz integral kernel  $H_\nu^p(t, x, x')$  of the weighted heat evolution operator with inverse square potential  $e^{tL_\nu}(\sqrt{-L_\nu})^p$  is given for  $\text{Re } p > -2(\nu + 1)$  as

$$H_\nu^p(t, x, x') = \frac{\Gamma(p/2 + 1 + \nu)}{[\Gamma(\nu + 1)]^2} \frac{(x/2)^{\nu+1/2} (x'/2)^{\nu+1/2}}{t^{p/2+\nu+1}} \Psi_2(p/2 + 1 + \nu, \nu + 1, \nu + 1; x^2/4t, x'^2/4t) \quad (2.1)$$

where  $\Psi_2(a, c, c'; x, y)$  denotes the Humbert's confluent hypergeometric function of two variables given in (1.15).

**Proof** Using the proposition 1.2 with  $\phi(\omega) = e^{-t\omega^2} \omega^p$  we have

$$H_\nu^p(t, x, x') = (xx')^{1/2} \int_0^\infty J_\nu(\omega x) J_\nu(\omega x') e^{-t\omega^2} \omega^{p+1} d\omega$$

next we employ the formula (1.14) and we arrive at the formula (2.1).

**Corollary 2.2** The Schwartz integral kernel  $K_\nu^p(t, x, x')$  of the weighted Schrödinger evolution operator with inverse square potential  $e^{itL_\nu}(\sqrt{-L_\nu})^p$  is given in terms of the two variables Humbert's confluent hypergeometric function for  $\text{Re } p > -2(\nu + 1)$  as

$$K_\nu^p(t, x, x') = \frac{\Gamma(p/2 + 1 + \nu)}{[\Gamma(\nu + 1)]^2} \frac{(x/2)^{\nu+1/2} (x'/2)^{\nu+1/2}}{(it)^{p/2+\nu+1}} \Psi_2(p/2 + 1 + \nu, \nu + 1, \nu + 1; x^2/4t, x'^2/4it) \quad (2.2)$$

### 3-Weighted wave evolution operator with inverse square potential on $\mathbb{R}^+$

In this section we shall compute explicitly the Schwartz integral kernel of the weighted wave evolution operators  $\frac{\sin t\sqrt{-L_\nu}}{\sqrt{-L_\nu}}(-L_\nu)^{p/2}$

**Theorem 3.1** For  $t > 0$  we have

$$\frac{\sin t\sqrt{-L_\nu}}{\sqrt{-L_\nu}} = \sqrt{\frac{\pi}{2}} \frac{t}{2i\pi} \int_{-\infty}^{0+} e^{(1/2)(u+t^2L_\nu/u)} u^{-3/2} du \quad (3.1)$$

Here we should note that the integral in (3.1) can be extended over a contour starting at  $\infty$ , going clockwise around 0, and returning back to  $\infty$  without cutting the real negative semi-axis.

**Proof** We start by recalling the formulas ([13], p.73)

$$\sin z = \sqrt{\pi z/2} J_{1/2}(z) \quad (3.2)$$

where  $J_\nu(\cdot)$  is the Bessel function of first kind and of order  $\nu$  given by ([5], p.83)

$$J_\nu(\alpha z) = \frac{z^\nu}{2i\pi} \int_{-\infty}^{0+} e^{(\alpha/2)(t-z^2/t)} t^{-\nu-1} dt \quad (3.3)$$

provided that  $\text{Re } \alpha > 0$  and  $|\arg z| \leq \pi$ . Moreover, we have the following formula:

$$\frac{\sin \alpha z}{z} = \sqrt{\frac{\pi}{2}} \frac{\sqrt{\alpha}}{2i\pi} \int_{-\infty}^{0+} e^{(\alpha/2)(u-z^2/u)} u^{-3/2} du \quad (3.4)$$

Putting  $\alpha = 1$  and replacing the variable  $z$  by the symbol  $t\sqrt{-L_\nu}$  in (3.4) we obtain the formula (3.1).

**Corollary 3.2** The Schwartz integral kernels  $H_\nu^p(t, x, x')$  and  $W_\nu^p(t, x, x')$  of the weighted heat and wave operators are related by the formula

$$W_\nu^p(t, x, x') = \frac{t}{2i\sqrt{2}\pi} \int_{-\infty}^{0+} e^{u/2} H_\nu^p(t^2/2u, x, x') u^{-3/2} du \quad (3.5)$$

**Proof** The results follow at once from the theorem 3.1 combined with the proposition 1.2.

**Corollary 3.3** We have the following formulas relating the modified Bessel function of order  $\nu$  and the Legendre functions of order  $\nu - 1/2$ .

$$\int_{-\infty}^{+0} e^{u/2} u^{-1/2} e^{u/2} e^{-\frac{x^2+x'^2}{2t^2}u} I_{\nu}\left(\frac{xx'}{t^2}u\right) du = \frac{2i\sqrt{2\pi}t}{\sqrt{xx'}} \begin{cases} 0 & x' > t+x \\ \frac{1}{2} P_{\nu-1/2}\left(\frac{x^2+x'^2-t^2}{2xx'}\right) & t-x < x' < t+x \\ \frac{\cos \pi \nu}{\pi} Q_{\nu-1/2}\left(\frac{t^2-x^2-x'^2}{2xx'}\right) & x' < t-x \end{cases} \quad (3.6)$$

**Proof** The formula (3.6) follows easily on using the corollary 3.2 with  $p = 0$  in conjunction with the formulas (1.6) and (1.7).

**Theorem 3.4** For  $-2 - 2\nu < p < 1$  and  $x + x' < t$ , the Schwartz integral kernel  $W_{\nu}^p(t, x, x')$  of the weighted wave evolution operator  $\frac{\sin t\sqrt{-L_{\nu}}}{\sqrt{-L_{\nu}}}(\sqrt{-L_{\nu}})^p$  is given the two following formulas

$$W_{\nu}^p(t, x, x') = \frac{2^{p-1}\Gamma(p/2+1+\nu)(xx')^{\nu+1/2}}{i\sqrt{\pi}[\Gamma(\nu+1)]^{2t^2+2\nu+1}} \times \int_{-\infty}^{0+} e^u u^{p/2+\nu-1/2} \Psi_2\left(p/2+1+\nu, \nu+1, \nu+1; \frac{x^2}{t^2}u, \frac{x'^2}{t^2}u\right) du \quad (3.7)$$

$$W_{\nu}^p(t, x, x') = \frac{2^p \sqrt{\pi} \Gamma(p/2+\nu+1)}{[\Gamma(\nu+1)]^{2t^2} \Gamma((1-p)/2-\nu)} \times \frac{(xx')^{\nu+1/2}}{t^{2\nu+p+1}} F_4\left(p/2+1/2+\nu, p/2+\nu+1, \nu+1, \nu+1; x^2/t^2, x'^2/t^2\right) \quad (3.8)$$

where  $\Psi_2$  the Humbert's confluent hypergeometric function in (1.15) and  $F_4$  is Appell's hypergeometric function given in (1.17).

**Proof.** Under the substitution of the value of  $H_{\nu}^p(t, x, x')$  given by (2.1) in the formula (3.5) we obtain the formula (3.7) by replacing the Humbert's confluent in (3.7) with its series expansion given by (1.15) and integrating term by term we get

$$W_{\nu}^p(t, x, x') = \frac{2^{p-1}\Gamma(p/2+1+\nu)(xx')^{\nu+1/2}}{i\sqrt{\pi}[\Gamma(\nu+1)]^{2t^2+2\nu+1}} \sum_{n,m \geq 0} \frac{(p/2+\nu+1)_{m+n}}{(\nu+1)_m(\nu+1)_n m! n!} \left(\frac{x^2}{t^2}\right)^m \left(\frac{x'^2}{t^2}\right)^n \int_{-\infty}^{0+} e^u u^{p/2+\nu-1/2+m+n} du \quad (3.9)$$

in view of the classical formulas for Euler  $\Gamma$  functions

$$\frac{1}{\Gamma(z)} = \frac{1}{2i\pi} \int_{\infty}^{0+} e^{t-z} dt \quad (3.10)$$

and

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \quad (3.11)$$

the formula (1.17) gives the formula (3.8).

**Corollaire 3.5** For  $x + x' < t$  we have

$$F_4\left(\nu+p/2+1/2, \nu+p/2+1, \nu+1, \nu+1; x^2/t^2, x'^2/t^2\right) = \frac{\Gamma((1-p)/2-\nu)}{2i\pi} \int_{-\infty}^{0+} e^u u^{(p-1)/2+\nu} \psi_2\left(p/2+\nu+1, \nu+1, \nu+1, \frac{x^2}{t^2}u, \frac{x'^2}{t^2}u\right) du \quad (3.12)$$

**Proof** Use corollary 3.4.

**Corollaire 3.6** For  $-2 - 2\nu < p < 1$  and  $x + x' < t$ , we have:

$$\int_0^{\infty} J_{\nu}(\omega x) J_{\nu}(\omega x') J_{1/2}(\omega t) \omega^{p+1/2} d\omega =$$

$$\frac{2^{p+1/2}\Gamma(p/2+\nu+1)}{[\Gamma(\nu+1)]^2\Gamma(-\nu+(1-p)/2)t^{2\nu+p+3/2}} (xx')^{\nu} F_4\left(\nu+p/2+1/2, \nu+p/2+1, \nu+1, \nu+1; x^2/t^2, x'^2/t^2\right) \quad (3.13)$$

**Proof** Using the formula (1.13) with

$$\phi(\omega) = \sin t\omega \omega^{p-1} = \sqrt{\pi t/2} J_{1/2}(t\omega) \omega^{p-1/2} \quad (3.14)$$

to obtain

$$W_{\nu}(t, x, x') = \sqrt{\pi t/2} (xx')^{1/2} \int_0^{\infty} J_{\nu}(\omega x) J_{\nu}(\omega x') J_{1/2}(\omega t) \omega^{p+1/2} d\omega \quad (3.15)$$

and the theorem 3.4 gives the results.

Note that the theorem 3.4 gives the Schwartz integral kernel of the weighted resolvent kernel only for the case  $x + x' < t$

which correspond to the domain of convergence of the  $F_4$ -hypergeometric double series and the problem remain open until now for the case  $|x - x'| < t < x + x'$ . The reason is that the behaviour of the hypergeometric double series  $F_4$  outside of the convergence region  $x + x' < t$  is not well-known. Note that the above integral involving three Bessels functions was evaluated by Bayley [1] only for the case  $x + x' < t$  (see [8] But for  $p = -1/2$  we can use a formula given in ([9], p.646 – 647) and to state the following results.

**Proposition 3.7** We have

$$\int_0^\infty J_\nu(\omega x) J_\nu(\omega x') J_{1/2}(\omega t) d\omega =$$

$$\frac{(xx')^\nu \Gamma(3/4 + \nu) \Gamma(1/4 + \nu)}{\pi [\Gamma(\nu + 1)]^2 t^{1+2\nu}} \begin{cases} \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} E(t, x, x', \nu) \right] & t < x - x', x > x' \\ \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} G(t, x, x', \nu) \right] & |x - x'| < t < x + x' \\ \cos(\pi(1/4 - \nu) E(t, x, -x', \nu)) & t > x + x' \end{cases} \quad (3.16)$$

with

$$E(t, x, x', \nu) = {}_2F_1 \left( \nu + 1/4, \nu + 3/4, \nu + 1; \frac{x}{t} e^{-u_{x'}} \right) {}_2F_1 \left( \nu + 1/4, \nu + 3/4, \nu + 1; -\frac{x'}{t} e^{-u_x} \right)$$

$$G(t, x, x', \nu) = {}_2F_1 \left( \nu + 1/4, \nu + 3/4, \nu + 1; \frac{x}{t} e^{-i\varphi_{x'}} \right) {}_2F_1 \left( \nu + 1/4, \nu + 3/4, \nu + 1; -\frac{x'}{t} e^{-i\varphi_x} \right)$$

$$x^2 = x'^2 + t^2 - 2x't \cosh u_x, x'^2 = x^2 + t^2 - 2xt \cosh u_{x'} \quad (3.17)$$

$$x^2 = x'^2 + t^2 - 2x't \cos \varphi_x, x'^2 = x^2 + t^2 - 2xt \cos \varphi_{x'} \quad (3.18)$$

**Theorem 3.8** The Schwartz integral kernel of the weighted wave operator  $\frac{\sin t\sqrt{-L_\nu}}{\sqrt{-L_\nu}}(\sqrt{-L_\nu})^{-1/2}$  is given by the following formulas

$$W_\nu^{-1/2}(t, x, x') = \frac{(xx')^{\nu+1/2} \Gamma(3/4+\nu) \Gamma(1/4+\nu)}{\sqrt{2\pi} [\Gamma(\nu+1)]^2 t^{1/2+2\nu}} \times$$

$$\begin{cases} \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} E(t, x, x', \nu) \right] & t < x - x', x > x' \\ \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} G(t, x, -x', \nu) \right] & |x - x'| < t < x + x' \\ \cos(\pi(1/4 - \nu) E(t, x, -x', \nu)) & t > x + x' \end{cases} \quad (3.19)$$

Note that for  $\nu = n + 3/4$  with  $n$  integer we have

$$W_{n+3/4}^{-1/2}(t, x, x') = (-1)^{n+1} \frac{(xx')^{n+5/4} \Gamma(3/2+n) \Gamma(n+1)}{\sqrt{2\pi} [\Gamma(n+7/4)]^2 t^{2(n+1)}} \times$$

$$\mathcal{I}m \left[ {}_2F_1 \left( n+1, n+3/2, n+7/4; \frac{x}{t} e^{-i\varphi_{x'}} \right) {}_2F_1 \left( n+1, n+3/2, n+7/4; \frac{x'}{t} e^{-i\varphi_x} \right) \right] 1_{\{|x-x'| < t < x+x'\}} \quad (3.20)$$

where  $\varphi_x$  and  $\varphi_{x'}$  are given in (3.6).

**Proof** We use the formulas ((3.15) and (3.16)).

#### 4- Weighted resolvent kernels

In this section we give explicit formula for the Schwartz integral kernel of the weighted resolvent operator

$$R_\nu^p(\lambda) = (L_\nu + \lambda^2)^{-1} (-L_\nu)^{p/2}.$$

We show first that one can write the weighted resolvent in terms of the weighted heat and the weighted wave kernels.

**Proposition 4 .1** Let  $R_\nu^p(\lambda, x, x')$  be the Schwartz kernel of the weighted resolvent operator then we have

$$R_\nu^p(\lambda, x, x') = \int_0^\infty e^{\lambda^2 t} H_\nu^p(t, x, x') dt; \quad \mathcal{R}e \lambda^2 < 0. \quad (4.1)$$

$$R_\nu^p(\lambda, x, x') = \int_0^\infty e^{-i\lambda t} W_\nu^p(t, x, x') dt \quad \mathcal{R}e \lambda^2 < 0 \quad (4.2)$$

where  $H_\nu^p(t, x, x')$  and  $W_\nu^p(t, x, x')$  are respectively the Schwartz integral kernels of the weighted heat and wave evolution operators.

**Proof** We use respectively the following formulas

$$(a^2 + y^2)^{-1} = \int_0^\infty e^{-(a^2+y^2)t} dt \quad \mathcal{R}e a > 0 \quad (4.3)$$

$$(a^2 + y^2)^{-1} = \int_0^\infty e^{-ax} \frac{\sin xy}{y} dx \quad \mathcal{R}e a > 0 \quad (4.4)$$

**Corollary 4.2** We have the following formulas

$$\frac{1}{2} \int_{x'-x}^{x+x'} e^{-i\lambda t} P_{\nu-1/2} \left( \frac{x^2 + x'^2 - t^2}{2xx'} \right) dt + \frac{\cos \pi \nu}{\pi} \int_{x+x'}^\infty e^{-i\lambda t} Q_{\nu-1/2} \left( \frac{t^2 - x^2 - x'^2}{2xx'} \right) dt =$$

$$\frac{i\pi\sqrt{xx'}}{2} \begin{cases} J_\nu(\lambda x)H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x')H_\nu^{(1)}(\lambda x) & x > x' \end{cases} \quad (4.5)$$

$$\int_0^\infty e^{\lambda^2 t} t^{-1} e^{-\frac{x^2+x'^2}{4t}} I_\nu\left(\frac{xx'}{2t}\right) dt = i\pi \begin{cases} J_\nu(\lambda x)H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x')H_\nu^{(1)}(\lambda x) & x > x' \end{cases} \quad (4.6)$$

**Proof** We use the proposition 4.1, (1.6), (1.7) and (1.8).

**Corollary 4.3** For  $\text{Im}\lambda^2 < 0$  we have

$$(xx')^{1/2} \int_0^\infty \frac{J_\nu(x\omega)J_\nu(x'\omega)}{-\omega^2 + \lambda^2} \omega d\omega = \frac{i\pi\sqrt{xx'}}{2} \begin{cases} J_\nu(\lambda x)H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x')H_\nu^{(1)}(\lambda x) & x > x' \end{cases} \quad (4.7)$$

**Proof** We use (1.13) with  $\phi(\omega) = (-\omega^2 + \lambda^2)^{-1}$  and the formula (1.8).

**Theorem 4.4** For  $\text{Im}\lambda^2 < 0$ , the Schwartz integral kernel for the weighted resolvent operator  $(L_\nu + \lambda^2)^{-1} (-L_\nu)^{p/2}$  is given by

$$R_\nu^p(\lambda, x, x') = \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu)}{[\Gamma(\nu + 1)]^2} (-\lambda^2)^{p/2 + \nu} (xx'/4)^{\nu+1/2} F_{0:1}^{0:0} \left( \begin{matrix} -:-:- \\ -:\nu+1;\nu+1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) \quad (4.8)$$

where  $F_{C:D}^{A:B}$  is the Kampé de Fériét generalized hypergeometric function given by [7]

$$F_{C:D}^{A:B} \left( \begin{matrix} a_1, \dots, a_A; b_1, \dots, b_B; b'_1, \dots, b'_B \\ c_1, \dots, c_C; d_1, \dots, d_D; d'_1, \dots, d'_D \end{matrix}; x, x' \right) = \sum_{n, m \geq 0} \frac{\prod_{j=1}^A (a_j)_{m+n} \prod_{j=1}^B (b_j)_m \prod_{j=1}^B (b'_j)_n}{\prod_{j=1}^C (c_j)_{m+n} \prod_{j=1}^D (d_j)_m \prod_{j=1}^D (d'_j)_n m! n!} x^m x'^n \quad (4.9)$$

**Proof** From the formula (4.1), (2.1) and (1.15) we can write

$$R_\nu^p(\lambda, x, x') = \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu)}{[\Gamma(\nu + 1)]^2} (xx'/4)^{\nu+1/2} (-\lambda^2)^{p/2 + \nu} \times \\ \sum_{m, n \geq 0} \frac{(p/2 + \nu + 1)_{m+n} \Gamma(-p/2 - \nu - m - n)}{(\nu + 1)_m (\nu + 1)_n m! n!} (\lambda^2 x^2/4)^m (\lambda^2 x'^2/4)^n$$

using the formula (3.8) we get result of the theorem.

**Corollary 4.5** We have the following formula

$$\int_0^\infty \frac{J_\nu(x\omega)J_\nu(x'\omega)}{-\omega^2 + \lambda^2} \omega^{p+1} d\omega = \\ \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu)}{2[\Gamma(\nu + 1)]^2} (-\lambda^2)^{p/2 + \nu} (xx'/4)^\nu F_{0:1}^{0:0} \left( \begin{matrix} -:-:- \\ -:\nu+1;\nu+1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) \quad (4.10)$$

**Proof** Using the formula (1.13) with  $\phi(\omega) = (-\omega^2 + \lambda^2)^{-1} \omega^p$  and the theorem 4.4.

**Corollary 4.6** The Schwartz integral kernel for the weighted resolvent operator  $(L_\nu + \lambda^2)^{-1} (\sqrt{-L_\nu})^{-1/2}$  is given by

$$R_\nu^{-1/2}(\lambda, x, x') = \frac{(xx')^{\nu+1/2} \Gamma(3/4 + \nu) \Gamma(1/4 + \nu)}{\sqrt{2\pi} [\Gamma(\nu + 1)]^2} \int_0^{x-x'} e^{-i\lambda t} t^{-1/2-2\nu} \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} E(t, x, x', \nu) \right] dt + \\ \int_{x-x'}^{x+x'} e^{-i\lambda t} t^{-1/2-2\nu} \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} G(t, x, x', \nu) \right] dt + \int_{x+x'}^\infty e^{-i\lambda t} t^{-1/2-2\nu} \cos(\pi(1/4-\nu)) E(t, x, -x', \nu) dt \quad (4.11)$$

and

$$R_{n+3/4}^{-1/2}(\lambda, x, x') = (-1)^{n+1} \frac{(xx')^{n+5/4} \Gamma(3/2 + n) \Gamma(n + 1)}{\sqrt{2\pi} [\Gamma(n + 7/4)]^2} \int_{|x-x'|}^{x+x'} e^{-i\lambda t} t^{-2(n+1)} \mathcal{I}m [G(t, x, -x', n + 1)] dt \quad (4.12)$$

**Proof** We use (4.2), (3.19), (3.20).

**Corollary 4.7** We have

$$F_{0:1}^{0:0} \left( \begin{matrix} -:-:- \\ -:\nu+1;\nu+1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) = \frac{(-\lambda^2)^{1/4-\nu} 2^{2\nu+1} \Gamma(1/4 + \nu)}{\sqrt{2\pi} \Gamma(-1/4 - \nu)} \int_0^{x-x'} e^{-i\lambda t} t^{-1/2-2\nu} \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} E(t, x, x', \nu) \right] dt + \\ \int_{x-x'}^{x+x'} e^{-i\lambda t} t^{-1/2-2\nu} \mathcal{R}e \left[ e^{-i\pi/2(1/2-2\nu)} G(t, x, -x', \nu) \right] dt + \int_{x+x'}^\infty e^{-i\lambda t} t^{-1/2-2\nu} \cos(\pi(1/4-\nu)) E(t, x, -x', \nu) dt \quad (4.13)$$

and for  $n$  integer we have

$$F_{0:1}^{0:0} \left( \begin{matrix} -:-:- \\ -:n+7/4;n+7/4 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) = \\ \frac{(-\lambda^2)^{-n-1/2} 2^{2n+5/2} \Gamma(n + 1)}{\sqrt{2\pi} \Gamma(-n - 1)} \int_{|x-x'|}^{x+x'} e^{-i\lambda t} t^{-2(n+1)} \mathcal{I}m [E(t, x, -x', n + 1)] dt \quad (4.14)$$

where  $F_{C:D}^{A:B}$  is the Kampé de Fériét generalized hypergeometric function given by [7].

**Corollary 4.8** We have the following formulas

$$R_\nu^p(\lambda, x, x') = \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu)}{\Gamma(\nu + 1)\Gamma(-\nu)} (-\lambda^2)^{p/2} R_\nu^0(\lambda, x, x') \quad (4.15)$$

and

$$R_\nu^p(\lambda, x, x') = \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu)}{\Gamma(\nu + 1)\Gamma(-\nu)} (-\lambda^2)^{p/2} \frac{i\pi}{2} \sqrt{xx'} \begin{cases} J_\nu(\lambda x) H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x') H_\nu^{(1)}(\lambda x) & x > x' \end{cases} \quad (4.16)$$

The proof of this Corollary is simple and in consequence is left to thereader.

### 5- Weighted generalized resolvent kernel

In this section we generalize some results of the section 4 by giving an explicit expression of the weighted generalized resolvent kernels  $R_\nu^{\mu,p}(\lambda) = (L_\nu + \lambda^2)^{-1-\mu} (-L_\nu)^{p/2}$ .

**Proposition 5.1** We have the following formula connecting the weighted generalized resolvent kernel to the weighted heat kernel

$$R_\nu^{\mu,p}(\lambda, x, x') = \frac{1}{\Gamma(\mu + 1)} \int_0^\infty e^{\lambda^2 t} t^\mu H_\nu^p(t, x, x') dt; \quad \operatorname{Re} \lambda^2 < 0. \quad (5.1)$$

**Proof** We use the formula

$$(a^2 + y^2)^{-1-\mu} = \frac{1}{\Gamma(\mu + 1)} \int_0^\infty e^{-(a^2 + y^2)t} t^\mu dt \quad \operatorname{Re} a > 0 \quad (5.2)$$

**Theorem 5.2** For  $\operatorname{Re} \lambda^2 < 0$  The Schwartz kernel of the weighted generalized resolvent kernel with inverse square potential is given by

$$R_\nu^{\mu,p}(\lambda, x, x') = \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu + \mu)}{[\Gamma(\nu + 1)]^2} \times (-\lambda^2)^{p/2 + \nu - \mu} (xx'/4)^{\nu + 1/2} F_{1:1}^{1:0} \left( \begin{matrix} p/2 + \nu + 1; - \\ p/2 + \nu + 1 - \mu; \nu + 1; \nu + 1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) \quad (5.3)$$

where  $F_{C:D}^{A:B}$  is the Kampé de Fériét generalized hypergeometric function given by 4.9.

**Proof** From the formula (5.1), (2.1) and (1.15) integrating term by term the formula (3.8) we arrive at the announced formula in (5.3)

**Corollary 5.3** We have the following formula

$$(xx')^{1/2} \int_0^\infty \frac{J_\nu(x\omega) J_\nu(x'\omega)}{(-\omega^2 + \lambda^2)^{1+\mu}} \omega^{p+1} d\omega = \frac{\Gamma(p/2 + \nu + 1)\Gamma(-p/2 - \nu + \mu)}{[\Gamma(\nu + 1)]^2} (-\lambda^2)^{p/2 + \nu - \mu} (xx'/4)^{\nu + 1/2} \times F_{1:1}^{1:0} \left( \begin{matrix} p/2 + \nu + 1; - \\ p/2 + \nu + 1 - \mu; \nu + 1; \nu + 1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) \quad (5.4)$$

**Proof** Using the formula (1.13) with  $p(\omega) = (-\omega^2 + \lambda^2)^{-1-\mu} \omega^p$  and the theorem 5.2.

**Corollary 5.4** We have the following formula

$$\begin{aligned} \int_0^\infty e^{\lambda^2 t} t^{\mu - p/2 - \nu - 1} \Psi_2(p/2 + 1 + \nu, \nu + 1, \nu + 1; x^2/4t, x'^2/4t) dt \\ = \Gamma(\mu + 1)\Gamma(-p/2 - \nu + \mu)(-\lambda)^{p/2 + \nu - \mu} F_{1:1}^{1:0} \left( \begin{matrix} p/2 + \nu + 1; - \\ p/2 + \nu + 1 - \mu; \nu + 1; \nu + 1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) \end{aligned} \quad (5.5)$$

**Proof** We use (5.1) with (2.1) and (5.3).

### 6- Applications

In this section we give some applications.

**Corollary 6.1**

$$\begin{aligned} \frac{1}{2} \int_{x'-x}^{x+x'} \sin \lambda t P_{\nu-1/2} \left( \frac{x^2 + x'^2 - t^2}{2xx'} \right) dt + \frac{\cos \pi \nu}{\pi} \int_{x+x'}^\infty \sin \lambda t Q_{\nu-1/2} \left( \frac{t^2 - x^2 - x'^2}{2xx'} \right) dt = \\ - \frac{\pi \sqrt{xx'}}{2} J_\nu(\lambda x) J_\nu(\lambda x') \end{aligned} \quad (6.1)$$

$$\frac{1}{2} \int_{x'-x}^{x+x'} \cos \lambda t P_{\nu-1/2} \left( \frac{x^2 + x'^2 - t^2}{2xx'} \right) dt + \frac{\cos \pi \nu}{\pi} \int_{x+x'}^\infty \cos \lambda t Q_{\nu-1/2} \left( \frac{t^2 - x^2 - x'^2}{2xx'} \right) dt =$$

$$\frac{i\pi\sqrt{xx'}}{2} \begin{cases} J_\nu(\lambda x)Y_\nu(\lambda x') & x < x' \\ J_\nu(\lambda x')Y_\nu(\lambda x) & x > x' \end{cases} \quad (6.2)$$

**Corollary 6.2** If  $\nu - 1/2 = n$  integer we have

$$\frac{1}{2} \int_{x'-x}^{x+x'} \sin \lambda t P_n \left( \frac{x^2 + x'^2 - t^2}{2xx'} \right) dt = -\frac{\pi\sqrt{xx'}}{2} J_{n-1/2}(\lambda x) J_{n-1/2}(\lambda x') \quad (6.3)$$

$$\frac{1}{2} \int_{x'-x}^{x+x'} \cos \lambda t P_n \left( \frac{x^2 + x'^2 - t^2}{2xx'} \right) dt = \frac{\pi\sqrt{xx'}}{2} \begin{cases} J_{\nu-1/2}(\lambda x) Y_{n-1/2}(\lambda x') & x < x' \\ J_{n-1/2}(\lambda x') Y_{\nu-1/2}(\lambda x) & x > x' \end{cases} \quad (6.4)$$

**Corollary 6.3** The heat kernel with an inverse square potential is given by the formula

$$H_\nu^0(t, x, x') = \frac{1}{[\Gamma(\nu+1)]} (r^2/4t)^{\nu/2} (\rho^2/4t)^{\nu/2} \frac{(r\rho)^{1/2}}{2t} \Psi_2(\nu+1, \nu+1, \nu+1; r^2/4t, \rho^2/4t) \quad (6.5)$$

and we have

$$\Psi_2(\nu+1, \nu+1, \nu+1; x, y) = \Gamma(\nu+1)(xy)^{-\nu/2} e^{-x-y} I_\nu(2\sqrt{xy}) \quad (6.6)$$

**Proof:** This can be seen from theorem 2.1 by taking  $p = 0$ .

**Proposition 6.4** We have

$$e^{tL_\nu} = \int_0^\infty \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} \frac{\sin s\sqrt{-L_\nu}}{\sqrt{-L_\nu}} s ds \quad (6.7)$$

**Proof** We use the formula

$$\int_0^\infty e^{-as^2} \frac{\sin us}{u} s ds = \frac{\sqrt{\pi}}{4a^{3/2}} e^{-\frac{u^2}{4a}}$$

**Corollary 6.5** We have

$$\begin{aligned} \frac{(xx')^{1/2}}{2t} e^{-\frac{(x^2+x'^2)}{4t}} I_\nu\left(\frac{xx'}{2t}\right) &= \frac{1}{2} \int_{x'-x}^{x+x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} P_{\nu-1/2}\left(\frac{x^2+x'^2-s^2}{2xx'}\right) s ds + \\ &\frac{\cos \pi \nu}{\pi} \int_{x+x'}^\infty \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} Q_{\nu-1/2}\left(\frac{s^2-x^2-x'^2}{2xx'}\right) s ds \end{aligned} \quad (6.8)$$

For  $\nu - 1/2 = n$  integer we have

$$\frac{(xx')^{1/2}}{2t} e^{-\frac{(x^2+x'^2)}{4t}} I_{n+1/2}\left(\frac{xx'}{2t}\right) = \frac{1}{2} \int_{x'-x}^{x+x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} P_n\left(\frac{x^2+x'^2-s^2}{2xx'}\right) s ds \quad (6.9)$$

**Proof** We combine the formulas (6.7), (1.6) and (1.7).

**Corollary 6.6** We have the following formula

The Schwartz integral kernel for the weighted resolvent operator  $(L_\nu + \lambda^2)^{-1} (\sqrt{-L_\nu})^{-1/2}$  is given by

$$\begin{aligned} H_\nu^{-1/2}(t, x, x') &= \frac{(xx')^{\nu+1/2} \Gamma(3/4+\nu) \Gamma(1/4+\nu)}{\sqrt{2\pi} [\Gamma(\nu+1)]^2} \int_0^{x-x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-1/2-2\nu} \mathcal{R} e^{-i\pi/2(1/2-2\nu)} E(t, x, x', \nu) s ds + \\ &\int_{x-x'}^{x+x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-1/2-2\nu} \mathcal{R} e^{-i\pi/2(1/2-2\nu)} G(t, x, -x', \nu) s ds + \int_{x+x'}^\infty \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-1/2-2\nu} \cos(\pi(1/4-\nu)) E(t, x, -x', \nu) s ds \end{aligned} \quad (6.10)$$

and

$$H_{n+3/4}^{-1/2}(t, x, x') = (-1)^{n+1} \frac{(xx')^{n+5/4} \Gamma(3/2+n) \Gamma(n+1)}{\sqrt{2\pi} [\Gamma(n+7/4)]^2} \int_{|x-x'|}^{x+x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-2(n+1)} \mathcal{I} m G(t, x, -x', n+1) s ds \quad (6.11)$$

**Corollary 6.7** We have the following formula

$$\begin{aligned} \Psi_2(p/2+1+\nu, \nu+1, \nu+1; x^2/4t, x'^2/4t) &= \\ &\frac{\sqrt{\pi}}{2^{2\nu+1/2} \Gamma(1/4+\nu) t^{3/4+\nu}} \left[ \int_0^{x-x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-1/2-2\nu} \mathcal{R} [e^{-i\pi/2(1/2-2\nu)} E(t, x, x', \nu) s ds + \right. \\ &\left. \int_{x-x'}^{x+x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-1/2-2\nu} \mathcal{R} e^{-i\pi/2(1/2-2\nu)} G(t, x, -x', \nu) s ds + \int_{x+x'}^\infty \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-1/2-2\nu} \cos(\pi(1/4-\nu)) E(t, x, -x', \nu) s ds \right] \end{aligned} \quad (6.12)$$

and

$$\Psi_2(3/2+n, 5/2+n, 5/2+n, x^2/4t, x'^2/4t) = (-1)^{n+1} \frac{(xx')^{n+5/4} \Gamma(3/2+n) \Gamma(n+1)}{\sqrt{2\pi} [\Gamma(n+7/4)]^2} \times$$

$$\int_{|x-x'|}^{x+x'} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{4\pi t^3}} s^{-2(n+1)} \mathcal{I} m [G(t, x, -x', n+1)] s ds \quad (6.13)$$



**Proof** Use the theorem 2.1 and the corollary 6.6.

**Corollary 6.8** We have the following formula

$$F_{0:1}^{0:0} \left( \begin{matrix} -; -; - \\ -; \nu+1; \nu+1 \end{matrix}; \frac{\lambda^2 x^2}{4}, \frac{\lambda^2 x'^2}{4} \right) = i\pi(-\lambda^2 x x' / 4)^{-\nu} \frac{\Gamma(\nu+1)}{\Gamma(-\nu)} \begin{cases} J_\nu(\lambda x) H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x') H_\nu^{(1)}(\lambda x) & x > x' \end{cases} \quad (6.14)$$

where  $F_{C:D}^{A:B}$  is the Kampé de Fériét generalized hypergeometric function given by [7]

## 7-Commentaries

The subject of study of this paper is situated at the meeting point of the partial differential equations, the integral transforms and the special functions of the mathematical physics.

Firstly explicit solutions of the following partial differential equations are given in terms of the hypergeometric functions. In fact these multipliers solvent the following heat, wave and Helmholtz equation with inverse square potential subject to the following conditions:

$$\begin{cases} \frac{\partial}{\partial t} u(t, x) - L_\nu u(t, x) = 0; (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^* \\ u(0, x) = (\sqrt{-L_\nu})^p u_0(x); u_0 \in C_0^\infty(\mathbb{R}^*) \end{cases} \quad (H_\nu^p)$$

$$\begin{cases} \frac{\partial^2}{\partial t^2} U(t, x) - L_\nu U(t, x) = 0; (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^* \\ U(0, x) = 0, U_t(0, x) = (\sqrt{-L_\nu})^p U_1(x); U_1 \in C^\infty(\mathbb{R}_+^*) \end{cases} \quad (W_\nu^p)$$

and

$$(L_\nu + \lambda^2) R(\lambda, x, x') = (\sqrt{-L_\nu})^p \delta_{x'}; x, x' \in \mathbb{R}_+^* \quad (R_\nu^p)$$

In fact here we give the solution of these Cauchy problems which is not dependent explicitly on the operator  $-L_\nu$ . Secondly some old and new formulas involving the hypergeometric functions are given by comparing these solutions.

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